

Chapter II: DIFFERENTIATION

Dr. Pham Huu Anh Ngoc
Department of Mathematics
International university
Saigon, Vietnam
Email: phangoc@hcmiu.edu.vn

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2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS



$$\frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

Example: Differentiate $y = x^2 \sin x$.

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS



Because $\sin x$ and $\cos x$ are differentiable, the following functions

$$\begin{array}{ll} \tan x = \frac{\sin x}{\cos x} & \sec x = \frac{1}{\cos x} \\ \cot x = \frac{\cos x}{\sin x} & \csc x = \frac{1}{\sin x} \end{array}$$

are differentiable. Their derivatives are

$$\begin{array}{ll} \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x & \frac{d}{dx}(\sec x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x \\ \frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x} & \frac{d}{dx}(\csc x) = -\csc x \cot x \end{array}$$

2.5 Derivatives of trigonometric functions

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How to prove that

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x.$$

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$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x.$$

Solution: Using the Quotient Rule, we have

$$\begin{aligned} \frac{d}{dx} (\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\ &= \frac{\cos x \frac{d}{dx} (\sin x) - \sin x \frac{d}{dx} (\cos x)}{\cos^2 x} \\ &= \frac{\cos x \cdot \cos x - \sin x (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.1 THE CHAIN RULE

Example: Evaluate F' where $F(x) = \sqrt{x^2 + 1}$.

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



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The differentiation formulas you learned in the previous sections do not enable you to calculate the derivative of this function.

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.1 THE CHAIN RULE

Example: Evaluate F' where $F(x) = \sqrt{x^2 + 1}$.

The differentiation formulas you learned in the previous sections do not enable you to calculate the derivative of this function.

Theorem 6.1: (CHAIN RULE) If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



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Example: Evaluate F' where $F(x) = \sqrt{x^2 + 1}$.

Solution: Let $f(u) := \sqrt{u}$ and let $u = g(x) = x^2 + 1$. Then $F = f \circ g$. We already know that

$$f'(u) = \frac{1}{2\sqrt{u}}, \quad u > 0; \quad g'(x) = 2x.$$

By the chain rule, we get

$$F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot (2x) = \frac{x}{\sqrt{x^2 + 1}}.$$

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.1 THE CHAIN RULE

In Leibniz notation,

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or

$$\frac{d}{dx} f(u) = f'(u) \cdot \frac{du}{dx}$$

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.1 THE CHAIN RULE

NOTE: In using the Chain Rule we work from the outside to the inside.
The formula

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

says that we differentiate the outer function [at the inner function] and then we multiply by the derivative of the inner function.

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION

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NOTE: In using the Chain Rule we work **from the outside to the inside**.
The formula

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

says that we differentiate **the outer function [at the inner function]** and then we **multiply by the derivative of the inner function**. That is,

$$\frac{d}{dx} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.1 THE CHAIN RULE

Example: Find $F'(x)$ if $F(x) = \sin(x^2 + 1)$.

Example: Find y' at $x = 0$ if $y = \cos\left(\frac{\pi}{2} - 3x\right)$.

Example: Find y' for

$$y = \left(\frac{1-x}{1+x^2}\right)^2.$$

Example: Find y' for $y = a^x$, $a > 0$.

2.6 Implicit differentiation



The functions we have worked with so far have all been given by equations of the form $y = f(x)$. A function in this form is said to be in **explicit form**. For example,

$$y(x) = x \sin x; \quad y(x) = \sqrt{x^2 + 1}, \quad y(x) = \tan x - x.$$

Sometimes practical problems will lead to equations in which the function y is not written explicitly in terms of the independent variable x ; for example, equations such as

$$x^2 y^3 - 6 = 5y^3 + x, \quad (1)$$

$$x^3 + y^3 = 6xy. \quad (2)$$

Equations (1) and (2) implicitly define y as several functions of x and the function y is said to be in **implicit form**.

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.2 IMPLICIT DIFFERENTIATION

Question: How to evaluate the derivative of implicit functions?

Fortunately, there is a simple technique based on the Chain Rule that you can use to find $\frac{dy}{dx}$ without solving equations for y .

This technique is called **implicit differentiation**.

2.6.2 IMPLICIT DIFFERENTIATION

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2.6.2 IMPLICIT DIFFERENTIATION



EXAMPLE: Let $x^2y^3 - 6 = 5y^3 + x$. Find $y'(x)$.

Solution: Consider y as a function of x and differentiate each term of the equation

$$x^2y^3 - 6 = 5y^3 + x,$$

with respect to x , we have

$$2xy^3 + 3x^2y^2y' = 15y^2y' + 1.$$

Now solve this equation for y' , to get

$$y' = \frac{2xy^3 - 1}{15y^2 - 3x^2y^2}.$$

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.2 IMPLICIT DIFFERENTIATION

Implicit Differentiation ■ Suppose an equation defines y implicitly as a differentiable function of x . To find $\frac{dy}{dx}$,

1. Differentiate both sides of the equation with respect to x . Remember that y is really a function of x and use the chain rule when differentiating terms containing y .
2. Solve the differentiated equation algebraically for $\frac{dy}{dx}$.

Example 6.5 Find an equation of the tangent line to the curve having equation

$$x^2 + y^2 - 3xy + 4 = 0$$

at the point $(2, 4)$.

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$$x^2 + y^2 - 3xy + 4 = 0$$

with respect to x , to get

$$2x + 2yy' - 3(y + xy') = 0.$$

Thus, we have $y' = \frac{-2x+3y}{2y-3x}$. In particular, we have $y'(2) = \frac{-2 \cdot 2 + 3 \cdot 4}{2 \cdot 4 - 3 \cdot 2} = 4$.

The equation of the tangent line to the curve at $(2, 4)$ is

$$y - 4 = 4(x - 2),$$

or equivalently, $y = 4x - 4$.

2.6 CHAIN RULE AND IMPLICIT DIFFERENTIATION



2.6.2 IMPLICIT DIFFERENTIATION

Example 6.6 Find $\frac{dy}{dx}$ where $\cos(xy) = \frac{x^2}{y}$.

Solution: Consider y as a function of x and differentiate both sides of the equation $\cos(xy) = \frac{x^2}{y}$, to get

$$-\sin(xy) \cdot (y + xy') = \frac{2xy - y'x^2}{y^2}.$$

Solving the last equation for y' , we get

$$y' = \frac{-y^3 \sin(xy) - 2xy}{-x^2 + xy^2 \sin(xy)}.$$

HOW TO FIND DERIVATIVES OF INVERSE FUNCTIONS?

Theorem 7.1 Assume that $f(x)$ is differentiable and one-to-one with the inverse $g(x) = f^{-1}(x)$. If a belongs to the domain of $g(x)$ and $f'(g(a)) \neq 0$, then $g'(a)$ exists and

$$g'(a) = \frac{1}{f'(g(a))}$$

In Leibniz notation,

$$\frac{df^{-1}}{dx} = \frac{1}{\frac{df}{dx}}$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\begin{aligned}\frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}, & -1 < x < 1 \\ \frac{d}{dx} \cos^{-1} x &= -\frac{1}{\sqrt{1-x^2}}, & -1 < x < 1\end{aligned}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}$$

How to prove these identities?

2.7 DERIVATIVES OF INVERSE FUNCTIONS



We give the proof of the first identity. Note that

$$y = \sin^{-1}x \quad \text{means} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x , we obtain

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geq 0$, since $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} (\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$$

Example 7.1 Calculate $f'(\frac{1}{2})$, where $f(x) = \arcsin(x^2)$.

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Solution: Using the chain rule, we get

$$f'(x) = \frac{1}{\sqrt{1-x^4}} \cdot 2x.$$

In particular, we have

$$f'(\frac{1}{2}) = \frac{1}{\sqrt{1-\frac{1}{2^4}}} = \frac{4}{\sqrt{15}}.$$

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Example 7.2 Differentiate $f(x) = x \arctan \sqrt{x}$.

DERIVATIVES OF EXPONENTIAL FUNCTIONS

$$\frac{dy}{dx}(e^x) = e^x \qquad \frac{dy}{dx}(a^x) = a^x \ln a \quad (a > 0)$$

DERIVATIVES OF LOGARITHMIC FUNCTIONS

$$\frac{dy}{dx}(\ln |x|) = \frac{1}{x} \qquad \frac{dy}{dx}(\log_a |x|) = \frac{1}{x \ln a} \quad (a > 0, a \neq 1)$$

Combining the above formulae and the Chain Rule, we get

$$\begin{array}{ll} \frac{dy}{dx}(e^u) = e^u \frac{du}{dx} & \frac{dy}{dx}(a^u) = a^u \ln a \frac{du}{dx} \\ \frac{dy}{dx}(\ln |u|) = \frac{1}{u} \frac{du}{dx} & \frac{dy}{dx}(\log_a |u|) = \frac{1}{u \ln a} \frac{du}{dx} \end{array}$$

Example 7.3 Find the derivative of

$$(a) f(x) = e^{\sin x} \quad (b) g(x) = 10^{\sqrt[3]{x}} \quad (c) h(x) = \log_2 \sqrt{x^2 + 1}.$$

2.7.4 LOGARITHMIC DIFFERENTIATION

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The main idea:

♠ When we need to differentiate a function which involves lots of exponents and multiplication, we can use logarithms and implicit differentiation to make them easier. ♠

This method is called **logarithmic differentiation**.

Example: Differentiate

$$y(x) = \frac{(x+1)^2(x+2)^2(x+4)^3}{(x-1)(x+5)}.$$

2.7.4 LOGARITHMIC DIFFERENTIATION

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Example: Differentiate

$$y(x) = \frac{(x+1)^2(x+2)^2(x+4)^3}{(x-1)(x+5)}.$$

Observation: This of course would be a very difficult derivative requiring the quotient rule, the product rule (multiple times) and the chain rule (multiple times).

2.7.4 LOGARITHMIC DIFFERENTIATION



However, taking logarithms of both sides, we get

$$\ln y = \ln \left(\frac{(x+1)^2(x+2)^2(x+4)^3}{(x-1)(x+5)} \right).$$

Using properties of logarithmic functions, we get

$$\ln y = 2\ln(x+1) + 2\ln(x+2) + 3\ln(x+4) - \ln(x-1) - \ln(x+5).$$

Now using implicit differentiation, we have

$$\frac{y'}{y} = \frac{2}{x+1} + \frac{2}{x+2} + \frac{3}{x+4} - \frac{1}{x-1} - \frac{1}{x+5}.$$

Thus,

$$\begin{aligned} y' &= y \left(\frac{2}{x+1} + \frac{2}{x+2} + \frac{3}{x+4} - \frac{1}{x-1} - \frac{1}{x+5} \right) \\ &= \frac{(x+1)^2(x+2)^2(x+4)^3}{(x-1)(x+5)} \left(\frac{2}{x+1} + \frac{2}{x+2} + \frac{3}{x+4} - \frac{1}{x-1} - \frac{1}{x+5} \right). \end{aligned}$$

Example: Let $g(x) = \sqrt[4]{\frac{x^2+1}{x^2-1}}$. Find $g'(x)$.

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Solution: Set $y(x) = g(x) = \sqrt[4]{\frac{x^2+1}{x^2-1}}$. Take logarithms of both sides of this equation, to get

$$\ln(y) = \ln\left(\sqrt[4]{\frac{x^2+1}{x^2-1}}\right) = \frac{1}{4}(\ln(x^2+1) - \ln(x^2-1))$$

Differentiating both sides with respect to x , we get

$$y' \frac{1}{y} = \frac{1}{4} \left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1} \right).$$

Thus

$$g'(x) = \frac{y}{4} \left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1} \right) = \frac{1}{4} \sqrt[4]{\frac{x^2+1}{x^2-1}} \left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1} \right).$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x .
3. Solve the resulting equation for y' .

Example: Let $y(x) = x^x$, $x > 0$. Find $y'(x)$.

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
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Example: Let $y(x) = x^x$, $x > 0$. Find $y'(x)$.

Solution: Taking logarithms of both sides of the equation $y(x) = x^x$, we get $\ln y = \ln x^x = x \ln x$. Differentiate both sides of the last equation

$$\frac{y'}{y} = \ln x + x \frac{1}{x} = \ln x + 1.$$

Thus, $y'(x) = y(\ln x + 1) = x^x(\ln x + 1)$.

Example 7.4 Find the derivative of

$$f(x) = (\sin x)^x, \quad 0 < x < \pi.$$

In general, to differentiate a function of the form

$$y = u(x)^{v(x)}$$

we take natural logarithms of both sides and differentiate implicitly.

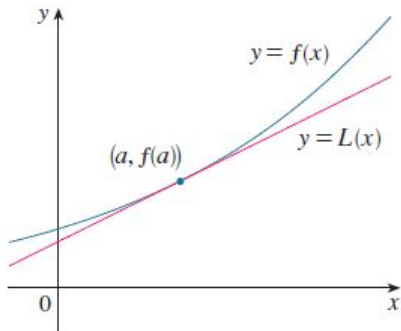
Example 7.5 Find the derivative of

$$y = \frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}.$$

2.8 LINEAR APPROXIMATION AND APPLICATIONS

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2.8.1 LINEAR APPROXIMATION



If x is close to a then

$$f(x) \approx L(x) := f(a) + f'(a)(x - a)$$

2.8 LINEAR APPROXIMATION AND APPLICATIONS



2.8.1 LINEAR APPROXIMATION

Definition 8.1 The approximation

$$f(x) \approx f(a) + f'(a)(x - a) \quad (0.1)$$

is called the **linear approximation** or **tangent line approximation** of f at $x = a$, and the function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at $x = a$.

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

SOLUTION The derivative of $f(x) = (x+3)^{1/2}$ is

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}$$

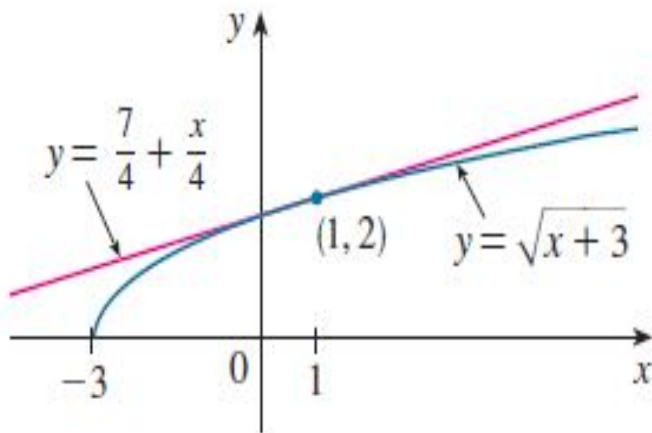
The corresponding linear approximation (1) is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

In particular, we have:

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \quad \text{and} \quad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$





Example: For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5?

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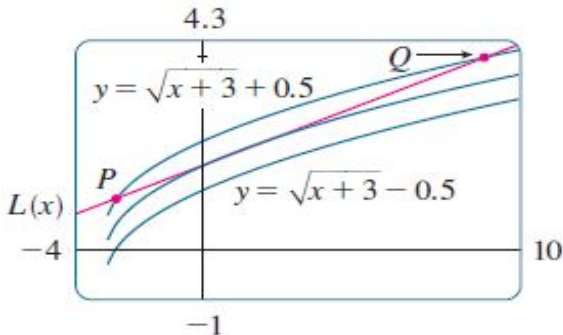
Solution: Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5.$$

This is equivalent to

$$\sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5.$$

This says that the linear approximation should lie between the curves obtained by shifting the curve $y(x) = \sqrt{x+3}$ upward and downward by an amount 0.5.



The tangent line intersects the upper curve at P and Q. The x-coordinate of P is about -2.66 and the x-coordinate of Q is about 8.66 . Thus, we see from the graph that the approximation $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ is accurate to within 0.5 when $-2.6 < x < 8.6$.

2.8 LINEAR APPROXIMATION AND APPLICATIONS



2.8.1 LINEAR APPROXIMATION

Recall that the **linear approximation** of f at $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a) \quad (0.2)$$

Let $\Delta x = x - a$ be the change in x and $\Delta y = f(x) - f(a)$ the corresponding change in y . Then (0.2) can be written as

$$\Delta y \approx f'(a)\Delta x.$$

2.8 LINEAR APPROXIMATION AND APPLICATIONS



2.8.1 LINEAR APPROXIMATION

Recall that the **linear approximation** of f at $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a) \quad (0.2)$$

Let $\Delta x = x - a$ be the change in x and $\Delta y = f(x) - f(a)$ the corresponding change in y . Then (0.2) can be written as

$$\Delta y \approx f'(a)\Delta x.$$

The product $f'(a)\Delta x$ is called the **differential** of f at a and denoted by df or dy :

$$dy = df = f'(a)\Delta x \quad (0.3)$$

In particular, if $f(x) = x$, then $df = dx = \Delta x$ at every point a .
Substituting $dx = \Delta x$ into Equation (0.3) we get

$$dy = df = f'(a)dx \quad (0.4)$$

We call dx the **differential of x** , and df the corresponding **differential of y** .

Linear approximation

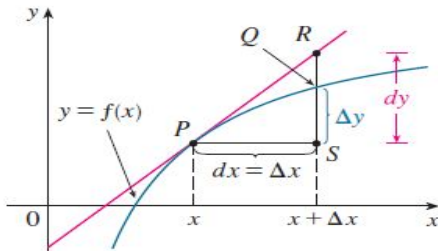
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Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x).$$

Therefore,

dy represents the amount that the tangent line rises or falls, whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes an amount dx .



2.8 LINEAR APPROXIMATION AND APPLICATIONS



2.8.2 DIFFERENTIALS

Since

$$\Delta y = f(x + \Delta x) - f(x) \approx dy,$$

we have

$$f(x + \Delta x) \approx f(x) + dy.$$

Example 8.2 The radius of a circle increases from an initial value of $r_0 = 10$ by an amount $dr = 0.1$. Estimate the corresponding increase in the circle's area $A = \pi r^2$ by calculating dA . Compare dA with the true change ΔA .

The **percentage error** is defined by

$$\text{percentage error} = \left| \frac{\text{error}}{\text{actual value}} \right| \times 100\%$$

2.8 LINEAR APPROXIMATION AND APPLICATIONS



2.8.3 THE SIZE OF THE ERROR

In any approximation, the error is defined by

$$\text{error} = |\text{true value} - \text{approximate value}|$$

If the linearization of $y = f(x)$ about $x = a$,

$$L(x) = f(a) + f'(a)(x - a),$$

is used to approximate $f(x)$ near a , then the error $E(x)$ in this approximation is

$$E(x) = |f(x) - L(x)| = |f(x) - f(a) - f'(a)(x - a)| = |\Delta y - dy|.$$

Theorem 8.1 (Error Bound) *If $|f''(x)| \leq K$ for all x in the interval between a and $a + h$, then*

$$E(x) \leq \frac{1}{2}Kh^2.$$

2.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES



Rates of change in Mathematics:

Definition 4.5 The **average rate of change** of a function $f(x)$ over the interval from x to $x + h$ is

$$\text{Average rate of change} = \frac{f(x + h) - f(x)}{h}$$

The **(instantaneous) rate of change** of f at x is the derivative:

$$\text{Rate of change of } f \text{ at } x = f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

provided that the limit exists.

2.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES



Example 4.3 Let $A = \pi r^2$ be the area of a circle of radius r .

- (a) Calculate the rate of change of area with respect to radius.
- (b) Compute dA/dr for $r = 2$ and $r = 5$, and explain why dA/dr is larger at $r = 5$.

2.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES



Example 4.4 Researchers have determined that the body mass of yearling bighorn sheep on Ram Mountain in Alberta, Canada, can be estimated by

$$M(t) = 27.5 + 0.3t - 0.001t^2,$$

where $M(t)$ is the mass of the sheep (in kg) and t is the number of days since May 25.

- (a) Find the average rate of change of the weight of a bighorn yearling between 70 and 75 days after May 25.
- (b) Find the (instantaneous) rate of change of weight for a yearling sheep whose age is 70 days past May 25.

2.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES



The Effect of a One-Unit Change

For small values of the increment h , the difference quotient is close to the derivative itself:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

2.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES



The Effect of a One-Unit Change

For small values of the increment h , the difference quotient is close to the derivative itself:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

In some applications, the approximation is already useful with $h = 1$:

$$f'(a) \approx f(a+1) - f(a)$$

In other words,

$f'(a)$ is approximately equal to the change in f caused by one-unit change in x when $x = a$.

2.4.1 Rates of Change in Physics



Average velocity, instantaneous velocity:

Suppose we have a body moving along a coordinate line and we know that its position at time t is $s = f(t)$.

In the interval from any time t to the slightly later time $t + \Delta t$, the body moves from position $s = f(t)$ to position $s + \Delta s = f(t + \Delta t)$. The body's net change in position, or **displacement**, for this short time interval is

$$\Delta s = f(t + \Delta t) - f(t).$$

Definition 4.1

If a body moves along a line from position $s = f(t)$ to position $s + \Delta s = f(t + \Delta t)$, the body's **average velocity** for the time interval from t to $t + \Delta t$ is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Definition 4.2

Instantaneous velocity (**velocity**) is the derivative of position with respect to time. If the position function of the body moving along a line is $s = f(t)$, the body's instantaneous velocity at time t is

$$v = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Motion under the Influence of Gravity

Galileo discovered that the height $s(t)$ and velocity $v(t)$ of an object tossed vertically in the air are given as function of time by the formulas

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2, \quad v(t) = \frac{ds}{dt} = v_0 - gt$$

The constants s_0 and v_0 are the *initial values*:

- $s_0 = s(0)$ is the position at time $t = 0$.
- $v_0 = v(0)$ is the velocity at time $t = 0$.
- g is the acceleration due to gravity on the surface of the earth, with value

$$g = 9.80 \text{ m/s}^2 = 32 \text{ ft/s}^2$$

- *The maximum height is attained when $v(t) = 0$.*

Example 4.1 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec. It reaches a height of $s = 160t - 16t^2$ ft after t sec.

- (a) How high does the rock go?
- (b) How fast is the rock traveling when it is 256 ft above the ground on the way up? on the way down?

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- (a) How high does the rock go?
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Solution: (a) At the highest point, the velocity of the rock will be 0:

$$v(t) = s'(t) = 160 - 32t$$

The velocity is zero, when $160 - 32t = 0$ or $t = 5$.

$$s(5) = 160.5 - 16.25 = 400ft.$$

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$$s(5) = 160(5) - 16(5)^2 = 400\text{ft}.$$

(b) Find t such that $s(t) = 160t - 16t^2 = 256$. Solve $16t^2 - 160t + 256 = 0$ for t , to get $t = 2$ and $t = 8$. Then,

$$v(2) = 160 - 32(2) = 96\text{ft/sec}; v(8) = 160 - 32(8) = -96\text{ft/sec}.$$

In both cases the rock's speed is 96ft/sec .

Speed

Definition 4.3 **Speed** is the magnitude of the velocity.

$$\text{speed} = |\text{velocity}|$$

Example 4.2 Suppose a person standing at the top of a building 112 feet high throws a ball vertically upward with an initial velocity of 90 ft/sec.

- (a) Find the ball's height and velocity at time t .
- (b) When does the ball hit the ground and what is its impact speed?
- (c) When is the velocity 0? What is the significance of this time?
- (b) How far does the ball travel during its flight?

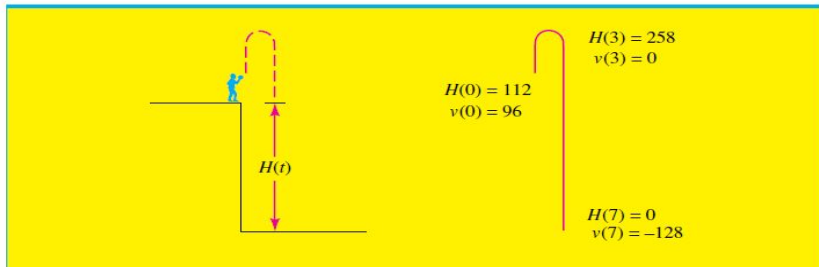


FIGURE 2.8 The motion of a ball thrown upward from the top of a building.

Solution

- (a) Since $g = 32$, $V_0 = 96$, and $H_0 = 112$, the height of the ball above the ground at time t is

$$H(t) = -16t^2 + 96t + 112 \text{ feet}$$

The velocity at time t is

$$v(t) = \frac{dH}{dt} = -32t + 96 \text{ (ft/sec)}$$

- (b) The ball hits the ground when $H = 0$. Solve the equation $-16t^2 + 96t + 112 = 0$ to find that this occurs when $t = 7$ and $t = -1$ (verify). Disregarding the negative time $t = -1$, which is not meaningful in this context, conclude that impact occurs when $t = 7$ seconds and that the impact velocity is

$$v(7) = -32(7) + 96 = -128 \text{ ft/sec}$$

(The negative sign means the ball is coming down at the moment of impact.)

- (c) The velocity is zero when $v(t) = -32t + 96 = 0$, which occurs when $t = 3$ seconds. For $t < 3$, the velocity is positive and the ball is rising, and for $t > 3$, the ball is falling (see Figure 2.8). Thus, the ball is at its highest point when $t = 3$ seconds.
- (d) The ball starts at $H(0) = 112$ feet and rises to a maximum height of $H(3) = 256$ feet before falling to the ground. Thus,

$$\text{Total distance traveled} = \underbrace{(256 - 112)}_{\text{up}} + \underbrace{256}_{\text{down}} = 400 \text{ feet}$$

Acceleration

In studies of motion, we usually assume that the body's position function $s = f(t)$ has a second derivative as well as a first. The first derivative gives the body's velocity as a function of time; the second derivative gives the body's acceleration.

Definition 4.4 **Acceleration** is the derivative of velocity:

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

So

acceleration is the second derivative of the position.

2.4.1 Rates of Change in Physics



If $a(t) > 0$, the velocity is increasing. This does not necessarily mean that the speed is increasing. The object is speeding up only when the velocity and acceleration *have the same sign*.

If velocity is	and acceleration is	then object is	and speed is
positive	positive	moving forward	increasing
positive	negative	moving forward	decreasing
negative	positive	moving backward	decreasing
negative	negative	moving backward	increasing

2.4.1 Rates of Change in Physics



We now consider some other rates of change in Physics.

A current exists whenever electric charges move. If ΔQ is the net charge that passes through a surface during a time period Δt , then the average current during this time interval is defined as:

$$\text{Average Current} = \frac{\Delta Q}{\Delta t} = \frac{Q - Q_0}{t - t_0}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** I at a given time t_0 :

$$I = \lim_{t \rightarrow t_0} \frac{Q - Q_0}{t - t_0} = \frac{dQ}{dt}$$

Thus, *the current is the rate at which charge flows through a surface.*

2.4.1 Rates of Change in Physics



If a given substance is kept at a constant temperature, then its volume V depends on its pressure P . The rate of change of volume with respect to pressure is the derivative dV/dP . As P increases, V decreases, so $dV/dP < 0$. The compressibility is defined by

$$\text{Isothermal Compressibility} = \beta = -\frac{1}{V} \frac{dV}{dP}$$

Velocity, compressibility, and current are not the only rates of change important in physics. Others include:

- Density
- Power (the rate at which work is done)
- Rate of heat flow
- Temperature gradient (the rate of change of temperature with respect to position)
- Rate of decay of a radioactive substance in nuclear physics



Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t . The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \rightarrow 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

2.4.3 Rates of Change in Biology



Example: Consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that **the population doubles every hour**. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0; \quad f(2) = 2f(1) = 2^2 n_0, \quad f(3) = 2f(2) = 2^3 n_0,$$

and, in general,

$$f(t) = 2^t n_0.$$

The population function is $n = 2^t n_0$. So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = 2^t (\ln 2) n_0 \approx (0.69) 2^t n_0.$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the **rate of growth after 4 hours** is

$$\left. \frac{dn}{dt} \right|_{t=4} \approx (0.69)2^4 100 = 1104.$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1100 bacteria per hour.

2.4.2 Rates of change in Economics



Marginal Cost: Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2 , then the additional cost is $\Delta C := C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Then the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{Marginal Cost} = \lim_{\Delta x \rightarrow 0} \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x} = C'(x_1).$$

2.4.2 Rates of change in Economics



If a company, currently producing x units, increases its production by one unit, then the additional cost ΔC of producing that one unit is

$$\Delta C = C(x+1) - C(x) \approx C'(x) \cdot 1 = C'(x).$$

The marginal cost is the change in total cost that arises when the quantity produced changes by one unit. That is, it is the cost of producing one more unit of a good.

2.4.2 Rates of change in Economics



Marginal Revenue and Marginal Profit Suppose $r(x)$ is the revenue generated when x units of a particular commodity are produced, and $p(x)$ is the corresponding profit. When $x = a$ units are being produced, then

- The marginal revenue is $r'(a)$. It approximates $r(a+1) - r(a)$, the additional revenue generated by producing one more unit.
- The marginal profit is $p'(a)$. It approximates $p(a+1) - p(a)$, the additional profit generated by producing one more unit.

Note

Revenue = (number of units sold) · (price per unit)

2.4.2 Rates of change in Economics



Example 4.5 *Marginal Cost.* Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x stoves and your shop is currently producing 10 stoves a day. About how much extra will it cost to produce one more stove a day?



2.4.2 Rates of change in Economics



Example 4.5 *Marginal Cost.* Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x stoves and your shop is currently producing 10 stoves a day. About how much extra will it cost to produce one more stove a day?

♠ The cost of producing one more stove a day when 10 are produced is about $c'(10)$. Since

$$c'(x) = (x^3 - 6x^2 + 15x)' = 3x^2 - 12x + 15,$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

Thus the additional cost will be about \$195 if you produce 11 stoves a day.

2.4.2 Rates of change in Economics



Example 4.6 *Marginal Revenue.* If

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x thousand candy bars, the marginal revenue when x thousand are sold is

$$r'(x) = (x^3 - 3x^2 + 12x)' = 3x^2 - 6x + 12.$$

The marginal revenue function **estimates** the increase in revenue that will **result from selling one additional unit**. If you currently sell 10 thousand candy bars a week, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 thousand bars a week.

2.4.2 Rates of change in Economics



Example 4.7 *Marginal Revenue and Profit.* A manufacturer estimates that when x units of a particular commodity are produced, the total cost will be $c(x) = \frac{1}{8}x^2 + 3x + 98$ dollars, and furthermore, that all x units will be sold when the price is $p(x) = \frac{1}{3}(75 - x)$ dollars per units.

- (a) Find the marginal cost and the marginal revenue.
- (b) Use marginal cost to estimate the cost of producing the ninth unit.
- (c) What is the actual cost of producing the ninth unit?
- (d) Use marginal revenue to estimate the revenue derived from the sale of the ninth unit.
- (e) What is the actual revenue from the sale of the ninth unit?

2.4 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES



2.4.2 OTHER RATES OF CHANGE

A SINGLE IDEA, MANY INTERPRETATIONS

Joseph Fourier (1768-1830):

“Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS



Derivative of the Sine

$$\frac{d}{dx}(\sin x) = \cos x$$

Derivative of the Cosine

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 5.1 A body hanging from a spring is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any latter time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t ?

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155-156	3, 5, 8, 21, 35	6, 22, 28, 31, 36
167-170	3, 21, 31	11, 30, 32, 41, 43
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204-205	9, 17, 27, 40,	4, 12, 15, 20, 21, 28, 30, 32, 34, 38
215-218	1, 3, 6,	2, 4, 5, 7, 10, 11, 14, 15, 16, 19, 22, 23

Pages	Exercises	Assignments
223-225	4, 12, 20, 24, 31, 33, 37	11, 17, 25, 26, 29, 32, 34
233-236	10, 16, 33, 42, 45, 54	19, 20, 22, 30, 32, 34, 37, 39, 40, 46, 47, 48, 49, 55 58, 60, 71, 74
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250-251	4, 6, 12, 28	11, 16, 19, 20, 23, 29, 33, 35, 38, 39
256-257	3, 6, 8	16, 18, 19